Auxetic behavior from rotating triangles

JOSEPH N. GRIMA*
Department of Chemistry, University of Malta, Msida, MSD 06, Malta
E-mail: joseph.grima@um.edu.mt

KENNETH E. EVANS
Department of Engineering, University of Exeter, North Park Road, Exeter EX4 4QF, UK
E-mail: K.E.Evans@ex.ac.uk

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Materials with a negative Poisson's ratio (auxetic) exhibit the very unusual property of becoming wider when stretched and narrower when compressed [1]. This property gives a material several beneficial effects such as increased shear stiffness, increased plane strain fracture toughness, increased indentation resistance and improved acoustic damping properties [1–5].

In recent years several auxetics have been manufactured by modifying the microstructure of existing materials, including foams [2, 6] and microporous polymers [7, 8]. A number of molecular-level auxetics have also been discovered, proposed or synthesised including nanostructured polymers [1, 9–12], metals [13], silicates [14, 15] and zeolites [12, 16, 17]. In all of these cases, one may observe that the negative Poisson's ratio results from the very particular nano or microstructure of the material (geometry) and the way this deforms when subjected to loads (the deformation mechanism).

A particularly interesting set of structures which exhibit negative Poisson's ratio may be constructed using squares or equilateral triangles connected together through simple hinges as illustrated in Fig. 1 [11, 12, 16–19]. It has been proposed that these systems exhibit negative Poisson's ratios if when loaded, they deform through a mechanism where the squares or triangles remain rigid but rotate (hinge) relative to each other. Whilst a considerable amount of modeling work has been performed on the 'rotating squares' system [12, 16–19], no attempt has been made as yet to derive the full compliance matrix **S** for the 'rotating triangles' structure which would give a full description of the mechanical properties of this system.

To address this we present the derivation of the compliance matrix **S** for the tessellated system built using hinged equilateral rigid triangles of length 'l' which are at an angle θ to each other (see Figs 1a and Fig. 2). Since this system is a planar structure, the compliance matrix **S** is a 3×3 matrix with components s_{ij} $(i, j = 1, 2, 3)$ defined by:

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix} = \mathbf{S} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{pmatrix} \quad (1)$$

where ε_1 and ε_2 are the strains in the Ox_1 and Ox_2 directions respectively, γ_{12} is the shear strain in the Ox_1–Ox_2 plane, σ_1 and σ_2 are the axial stresses in the Ox_1 and Ox_2 directions respectively, and τ_{12} is the shear stress in the Ox_1–Ox_2 plane.

Referring to Fig. 2, the simplest unit cell for this 'rotating triangles' tessellation is a parallelogram such as ABCD where the points A, B, C, and D are all tessellates of each other. This unit cell ABCD contains two triangles (AEF and DEG) at an angle θ ($= \angle$FED) to each other. It can be shown that for proper tessellation, all the angles in the structure (including those which are not tessellates of $\angle$FED, such as $\angle$EBG and $\angle$ABI) must also be equal to θ (see Note 1). Furthermore, it may also be shown that for all values of θ, ABCD is constrained to assume the shape of a rhombus with side lengths L and constant internal angles of $\pi/3$ and $2\pi/3$ (see Note 2) where:

$$L = \sqrt{2}l\sqrt{1 - \cos\left(\frac{\pi}{3} + \theta\right)} \quad (2)$$

Because of this, the 'rotating triangles' structure is constrained not to shear, i.e., the compliance matrix defined in Equation 1 will reduce to:

$$\mathbf{S} = \begin{pmatrix} s_{11} & s_{12} & 0 \\ s_{21} & s_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{E_1} & \frac{-\nu_{21}}{E_2} & 0 \\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3)$$

where E_1 and E_2 are the Young's moduli for loading in the Ox_1 and Ox_2 directions, respectively and ν_{12} and ν_{21}

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* Author to whom all correspondence should be addressed.

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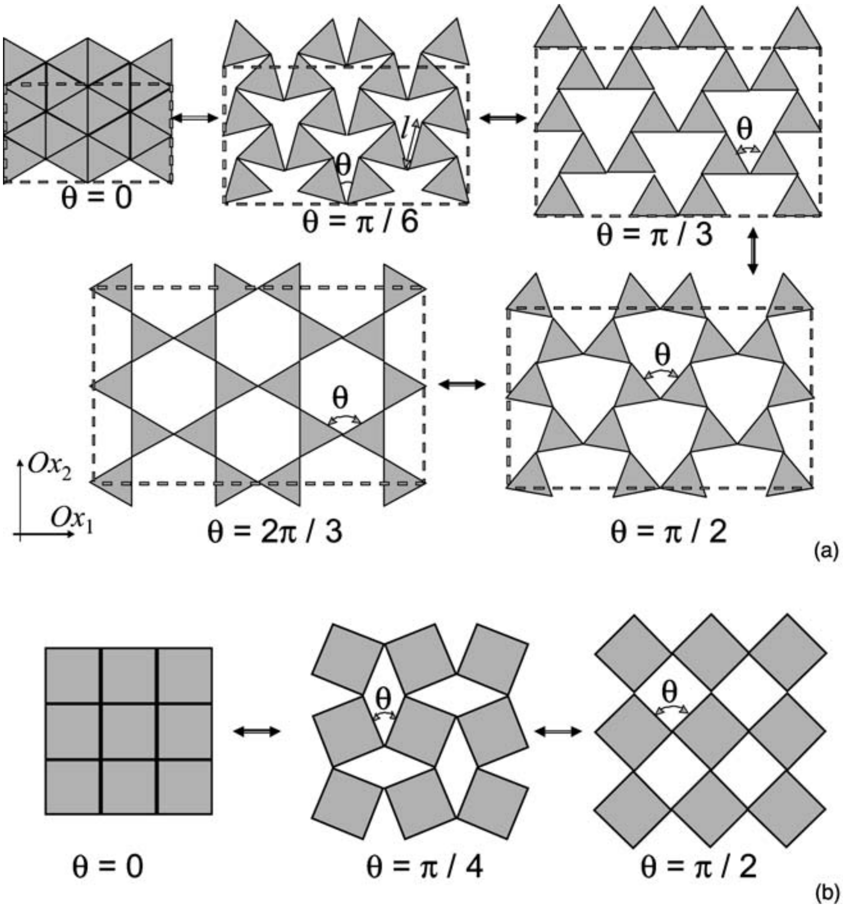

Figure 1 The geometry of the auxetic (a) 'rotating triangles' and (b) 'rotating squares' structures.

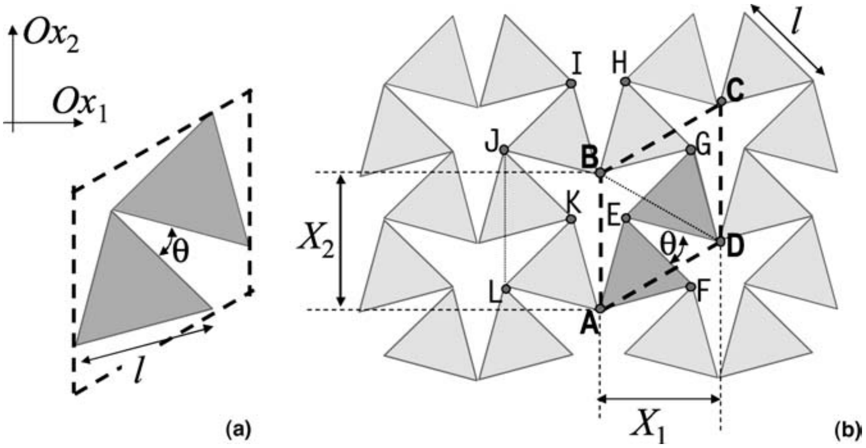

Figure 2 (a) The definition of a typical repeat unit and unit cell, and (b) the definition of the geometric parameters.

are the Poisson's ratios in Ox_1–Ox_2 plane for loading in the Ox_1 and Ox_2 directions, respectively.

To derive the Young's moduli and Poisson's ratios, one must consider the projections of the unit cell in the Ox_1 and Ox_2 directions which are given by:

$$X_1 = L \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}L \qquad (4)$$

$$X_2 = L \qquad (5)$$

In general, the Poisson's ratio is not constant and varies with strain and the initial geometry parameters and the strain dependent Poisson's functions ν_{12} and ν_{21} for loading in the Ox_1 and Ox_2 directions, respectively may be defined by:

$$\nu_{12} = -\frac{\mathrm{d}\varepsilon_2}{\mathrm{d}\varepsilon_1} \quad \textit{and} \quad \nu_{21} = -\frac{\mathrm{d}\varepsilon_1}{\mathrm{d}\varepsilon_2} \qquad (6)$$

where $\mathrm{d}\varepsilon_1$ and $\mathrm{d}\varepsilon_2$ are infinitesimally small strains in the Ox_1 and Ox_2 directions respectively which are given in terms of the unit cell dimensions by:

$$d\varepsilon_1 = \frac{\mathrm{d}X_1}{X_1} \quad \text{and} \quad \mathrm{d}\varepsilon_2 = \frac{\mathrm{d}X_2}{X_2} \tag{7}$$

where $\mathrm{d}X_1$ and $\mathrm{d}X_2$ are infinitesimally small changes in the unit cell dimensions X_1 and X_2, respectively due to the applied load. Thus, assuming the triangles do not deform upon loading, Equations 4 and 5 for X_1 and X_2 may be treated as functions of the single variable θ and hence the strains are given by:

$$d\varepsilon_1 = \frac{1}{X_1}\frac{\mathrm{d}X_1}{\mathrm{d}\theta}\mathrm{d}\theta \quad \text{and} \quad \mathrm{d}\varepsilon_2 = \frac{1}{X_2}\frac{\mathrm{d}X_2}{\mathrm{d}\theta}\mathrm{d}\theta \tag{8}$$

i.e.:

$$\mathrm{d}\varepsilon_1 = \frac{1}{X_1}\frac{\mathrm{d}X_1}{\mathrm{d}\theta}\mathrm{d}\theta = \left(\frac{\sqrt{3}}{2}L\right)^{-1}\left(\frac{\sqrt{3}}{2}L'\right)\mathrm{d}\theta = \frac{L'}{L}\mathrm{d}\theta \tag{9}$$

and

$$\mathrm{d}\varepsilon_2 = \frac{1}{X_2}\frac{\mathrm{d}X_2}{\mathrm{d}\theta}\mathrm{d}\theta = \left(\frac{1}{L}\right)(L')\mathrm{d}\theta = \frac{L'}{L}\mathrm{d}\theta \tag{10}$$

where:

$$L' = \frac{\mathrm{d}L}{\mathrm{d}\theta} = \frac{1}{\sqrt{2}}l\,\sin\left(\frac{\pi}{3}+\theta\right)\left[1-\cos\left(\frac{\pi}{3}+\theta\right)\right]^{-\frac{1}{2}} \tag{11}$$

Thus, from Equations 7–11, the on-axis Poisson's ratios are given by:

$$\nu_{21} = (\nu_{12})^{-1} = -\frac{\mathrm{d}\varepsilon_1}{\mathrm{d}\varepsilon_2} = -1 \tag{12}$$

The same result could have been trivially deduced from Note 2 as systems where the unit cells are constrained to be rhombic with constant cell angles will maintain their aspect ratio when stretched, i.e. exhibit Poisson's ratios of -1.

To derive the Young's moduli of the structure we shall assume that the stiffness in the structure is due to the stiffness of the hinges, defined by a stiffness constant K_h. The Young's moduli E_1 and E_2 may then be derived through a conservation of energy approach. In a continuum, the strain energies due to small strains $\mathrm{d}\varepsilon_1$ and $\mathrm{d}\varepsilon_2$ in the Ox_1 and Ox_2 directions, respectively are given by:

$$U = \frac{1}{2}E_1(\mathrm{d}\varepsilon_1)^2 \quad \text{and} \quad U = \frac{1}{2}E_2(\mathrm{d}\varepsilon_2)^2 \tag{13}$$

The strains $\mathrm{d}\varepsilon_1$ or $\mathrm{d}\varepsilon_2$ will result in a change in the angles θ. The work done per unit cell corresponding to this change in θ by '$\mathrm{d}\theta$' is given by:

$$W = N\left[\frac{1}{2}K_\mathrm{h}(d\theta)^2\right] \tag{14}$$

where N is the number of hinges corresponding to one unit cell. Since each unit cell contains two triangles, each triangle contains three vertices, and two vertices correspond to one hinge, then in this case, $N = 3$.

From the principle of conservation of energy Equations 13 and 14 are related through:

$$U = \frac{1}{V}W \tag{15}$$

where V is the volume of the unit cell. Assuming a unit thickness in the third dimension, this volume is given by:

$$V = X_1X_2 1 = \frac{\sqrt{3}}{2}L^2 \tag{16}$$

Thus, by combining Equations 4, 5, 9–11 and 13–16 we obtain:

$$E_1 = K_\mathrm{h}\frac{3X_1}{X_2}\left(\frac{\mathrm{d}X_1}{\mathrm{d}\theta}\right)^{-2} \quad \text{and } E_2 = K_\mathrm{h}\frac{3X_2}{X_1}\left(\frac{\mathrm{d}X_2}{\mathrm{d}\theta}\right)^{-2} \tag{17}$$

which simplifies to:

$$E_1 = E_2 = K_\mathrm{h}2\sqrt{3}\left(\frac{1}{L'}\right)^2 = K_\mathrm{h}\frac{4\sqrt{3}}{l^2\left[1+\cos\left(\frac{\pi}{3}+\theta\right)\right]} \tag{18}$$

This equation suggests that as θ increases from zero, it approaches $+\infty$ as $\theta \to 2\pi/3$ which corresponds to the point where the structure is fully extended.

The compliance matrix for this structure can hence be obtained by combining Equations 3, 12 and 18 to obtain:

$$\mathbf{S} = \begin{pmatrix} s_{11} & s_{12} & 0 \\ s_{21} & s_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{E_1} & \frac{-\nu_{21}}{E_2} & 0 \\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{E}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{19}$$

where $E = E_1 = E_2$. The off-axis mechanical properties obtained from the standard transformation equations [20] show that this idealised system is isotropic. This is particularly significant as it shows that in this highly idealised scenario where the triangles are equilateral and perfectly rigid, the Poisson's ratio will always assume a constant value of -1 irrespective of the size of the triangles, the angles between the triangles and the direction of loading. The fact that the Poisson's ratios are independent of the size of the triangles suggests that that this effect can be implemented at any scale, including at the molecular level where the 'rotating triangles' geometry is built into the nanostructure of the material. For example, it has been shown through force-field based simulations that a system made from 'graphite-like' layers each containing a planar polyphenylacetylene networked polymer can achieve Poisson's ratios close to the -1 by mimicking the 'rotating triangles' deformation mechanism [11, 12]. Furthermore, at the molecular level, the 'rotating triangles' geometry could be present as a projection of a particular plane in a three dimensional structure. This type of behavior is thought to be present in the zeolites ABW and JBW which have been predicted to exhibit negative Poisson's ratios in the planes containing a 'rotating triangles' projection [12, 17].

In real materials such as the one described here, deformations of the triangles will inevitably occur in parallel with the rotations and this may reduce the extent of the auxetic effect and allows the system to shear. The determining factor whether a material with a 'rotating triangles geometry' exhibits conventional or auxetic behavior would depend on which of the two deformation mechanisms dominates. In the cases described above, the force-field based simulations suggest that auxetic behavior is retained and this is very significant as it illustrates that the 'rotating triangles' mechanism can be a very effective way for introducing negative Poisson's ratios in real materials. We hope that these observations will encourage further research into the subject and will eventually lead the discovery of new materials with negative Poisson's ratios.

Notes:

Note 1: Proof that for proper tessellation, all the angles between the triangles are constrained to be of equal magnitude θ:

Assume that the system is constructed by tessellating the two triangles AEF and DEG which are at an angle $\angle\text{FED} = \theta$ to each other. For any values of l and θ, the lines EF and HG are parallel to each other (HG is a tessellate of EF) and hence $\angle\text{FEG} = \angle\text{EGH}$ (alternate angles). But, $\angle\text{FEG} = \angle\text{FED} + \angle\text{DEG}$ and $\angle\text{EGH} = \angle\text{EGB} + \angle\text{BGH}$ where $\angle\text{DEG} = \angle\text{BGH} = \frac{\pi}{3}$. Thus, $\angle\text{FED} = \angle\text{EGB} = \theta$. Similarly, it can be shown that since DG is parallel to BI, the angles $\angle\text{EGB} = \angle\text{HBI} = \theta$, etc.

Note 2: Proof that unit cell of the system is always a rhombus side lengths L and internal angles of $\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

For any values of l and θ:

1. Lengths $|\text{AB}| = |\text{DC}|$ and $|\text{BC}| = |\text{AD}|$ as these lines are tessellates of each other.

2. The lines LA and JB are tessellates and hence they have equal length and are parallel to each other. Thus, quadrilateral LJBA is a parallelogram and hence $|\text{LJ}| = |\text{AB}|$.

3. The line BC is the base of the isosceles triangle BHC with equal sides BH and HC of length l at an angle $\frac{\pi}{3} + \theta$ to each other. Thus, the length of line BC is given by $L = \sqrt{l^2 + l^2 - 2\,l\,l\,\cos(\frac{\pi}{3} + \theta)} = \sqrt{2}l\sqrt{1 - \cos(\frac{\pi}{3} + \theta)}$. Similarly, for LJ and BD which are the bases of triangles LKJ and BGD which are congruent to each other and to triangle BCH. Hence, $|\text{LJ}| = |\text{BD}| = |\text{BC}| = L$.

4. Thus, from (i), (ii) and (iii), $|\text{LJ}| = |\text{AB}| = |\text{BC}| = |\text{CD}| = |\text{DA}| = |\text{BD}| = L$ where $L = \sqrt{2}l\sqrt{1 - \cos(\frac{\pi}{3} + \theta)}$. This suggests that triangles ABD and DBC are equilateral for all values of θ and hence parallelogram ABCD is always a rhombus with side lengths L and internal angles of $\pi/3$ and $2\pi/3$.

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